Appendice A

Tabella di trasformate di Laplace

Per eseguire le operazioni di trasformazione e antitrasformazione secondo Laplace sono di notevole utilità tabelle che riportino le trasformate più importanti. Una di tali tabelle è riprodotta in questa appendice. La funzione $\arg(\sigma+j\omega)$ che in essa spesso ricorre è definita dalla relazione

$$\arg(\sigma+j\omega) := \begin{cases} \arctan\frac{\omega}{\sigma} & \text{per } \sigma \geq 0\ , \\ \pi + \arctan\frac{\omega}{\sigma} & \text{per } \sigma < 0,\ \omega \geq 0\ , \\ -\pi + \arctan\frac{\omega}{\sigma} & \text{per } \sigma < 0,\ \omega < 0\ ; \end{cases} \tag{A.1}$$

se la funzione viene programmata su un calcolatore, per non avere divisioni per zero (quando è σ=0), conviene usare l'espressione equivalente

$$\arg(\sigma+j\omega) := \begin{cases} \arcsin\frac{\omega}{\sqrt{\sigma^2+\omega^2}} & \text{per } \sigma \geq 0\ , \\ \pi - \arcsin\frac{\omega}{\sqrt{\sigma^2+\omega^2}} & \text{per } \sigma < 0,\ \omega \geq 0\ , \\ -\pi - \arcsin\frac{\omega}{\sqrt{\sigma^2+\omega^2}} & \text{per } \sigma < 0,\ \omega < 0\ . \end{cases} \tag{A.2}$$

	$F(s)$	$f(t)$
1.	1	$\delta(t)$ (impulso unitario in $t=0$)
2.	$\frac{1}{s}$	$u(t)$ (gradino unitario in $t=0$)
3.	$\frac{1}{s^2}$	$t\,u(t)$ (funzione rampa)

	$F(s)$	$f(t)$
4.	$\frac{1}{s^n}$	$\frac{1}{(n-1)!}t^{n-1}$
5.	$\frac{1}{s}e^{-as}$	$u(t-a)$ (gradino unitario in $t=a$)
6.	$\frac{1}{s}(1-e^{-as})$	$u(t)-u(t-a)$ (impulso rettangolare)
7.	$\frac{1}{s+a}$	e^{-at}
7a.	$\frac{1}{1+\tau s}$	$\frac{1}{\tau}e^{-\frac{t}{\tau}}$
8.	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8a.	$\frac{1}{(1+\tau s)^n}$	$\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-\frac{t}{\tau}}$
9.	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$
9a.	$\frac{1}{s(1+\tau s)}$	$1-e^{-\frac{t}{\tau}}$
10.	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$
10a.	$\frac{1}{(1+\tau_1 s)(1+\tau_2 s)}$	$\frac{1}{\tau_1-\tau_2}\left(e^{-\frac{t}{\tau_1}}-e^{-\frac{t}{\tau_2}}\right)$
11.	$\frac{s+\alpha}{(s+a)(s+b)}$	$\frac{1}{b-a}\left((\alpha-a)e^{-at}-(\alpha-b)e^{-bt}\right)$
11a.	$\frac{1+Ts}{(1+\tau_1 s)(1+\tau_2 s)}$	$\frac{1}{\tau_1-\tau_2}\left(\left(1-\frac{T}{\tau_1}\right)e^{-\frac{t}{\tau_1}}-\left(1-\frac{T}{\tau_2}\right)e^{-\frac{t}{\tau_2}}\right)$
12.	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left(1-\frac{b}{b-a}e^{-at}-\frac{a}{a-b}e^{-bt}\right)$
12a.	$\frac{1}{s(1+\tau_1 s)(1+\tau_2 s)}$	$1-\frac{\tau_1}{\tau_1-\tau_2}e^{-\frac{t}{\tau_1}}-\frac{\tau_2}{\tau_2-\tau_1}e^{-\frac{t}{\tau_2}}$
13.	$\frac{s+\alpha}{s(s+a)(s+b)}$	$\frac{1}{ab}\left(\alpha-\frac{b(\alpha-a)}{b-a}e^{-at}-\frac{a(\alpha-b)}{a-b}e^{-bt}\right)$
13a.	$\frac{1+Ts}{s(1+\tau_1 s)(1+\tau_2 s)}$	$1-\frac{\tau_1-T}{\tau_1-\tau_2}e^{-\frac{t}{\tau_1}}-\frac{\tau_2-T}{\tau_2-\tau_1}e^{-\frac{t}{\tau_2}}$
14.	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{1}{(b-a)(c-a)}e^{-at}+\frac{1}{(c-b)(a-b)}e^{-bt}+\frac{1}{(a-c)(b-c)}e^{-ct}$

	$F(s)$	$f(t)$
14a.	$\frac{1}{(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$	$\frac{\tau_1}{(\tau_1-\tau_2)(\tau_1-\tau_3)}e^{-\frac{t}{\tau_1}}+\frac{\tau_2}{(\tau_2-\tau_3)(\tau_2-\tau_1)}e^{-\frac{t}{\tau_2}}+\frac{\tau_3}{(\tau_3-\tau_1)(\tau_3-\tau_2)}e^{-\frac{t}{\tau_3}}$
15.	$\frac{s+\alpha}{(s+a)(s+b)(s+c)}$	$\frac{\alpha-a}{(b-a)(c-a)}e^{-at}+\frac{\alpha-b}{(c-b)(a-b)}e^{-bt}+\frac{\alpha-c}{(a-c)(b-c)}e^{-ct}$
15a.	$\frac{1+Ts}{(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$	$\frac{\tau_1-T}{(\tau_1-\tau_2)(\tau_1-\tau_3)}e^{-\frac{t}{\tau_1}}+\frac{\tau_2-T}{(\tau_2-\tau_3)(\tau_2-\tau_1)}e^{-\frac{t}{\tau_2}}+\frac{\tau_3-T}{(\tau_3-\tau_1)(\tau_3-\tau_2)}e^{-\frac{t}{\tau_3}}$
16.	$\frac{1}{s(s+a)(s+b)(s+c)}$	$\frac{1}{abc}-\frac{1}{a(b-a)(c-a)}e^{-at}-\frac{1}{b(c-b)(a-b)}e^{-bt}-\frac{1}{c(a-c)(b-c)}e^{-ct}$
16a.	$\frac{1}{s(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$	$1-\frac{\tau_1^2}{(\tau_1-\tau_2)(\tau_1-\tau_3)}e^{-\frac{t}{\tau_1}}-\frac{\tau_2^2}{(\tau_2-\tau_3)(\tau_2-\tau_1)}e^{-\frac{t}{\tau_2}}-\frac{\tau_3^2}{(\tau_3-\tau_1)(\tau_3-\tau_2)}e^{-\frac{t}{\tau_3}}$
17.	$\frac{s+\alpha}{s(s+a)(s+b)(s+c)}$	$\frac{\alpha}{abc}-\frac{\alpha-a}{a(b-a)(c-a)}e^{-at}-\frac{\alpha-b}{b(c-b)(a-b)}e^{-bt}-\frac{\alpha-c}{c(a-c)(b-c)}e^{-ct}$
17a.	$\frac{1+Ts}{s(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$	$1-\frac{\tau_1(\tau_1-T)}{(\tau_1-\tau_2)(\tau_1-\tau_3)}e^{-\frac{t}{\tau_1}}-\frac{\tau_2(\tau_2-T)}{(\tau_2-\tau_3)(\tau_2-\tau_1)}e^{-\frac{t}{\tau_2}}-\frac{\tau_3(\tau_3-T)}{(\tau_3-\tau_1)(\tau_3-\tau_2)}e^{-\frac{t}{\tau_3}}$
18.	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$
19.	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$
20.	$\frac{\omega\cos\varphi \pm s\sin\varphi}{s^2+\omega^2}$	$\sin(\omega t \pm \varphi)$
21.	$\frac{s\cos\varphi \mp \omega\sin\varphi}{s^2+\omega^2}$	$\cos(\omega t \pm \varphi)$
22.	$\frac{s+\alpha}{s^2+\omega^2}$	$\frac{\sqrt{\alpha^2+\omega^2}}{\omega}\sin(\omega t+\varphi)$, $\varphi := \arg(\alpha+j\omega)$
22a.	$\frac{\omega_n^2(1+Ts)}{s^2+\omega_n^2}$	$\omega_n\sqrt{1+\omega_n^2T^2}\sin(\omega_n t+\varphi)$, $\varphi := \arg(1+j\omega_n T)$

	$F(s)$	$f(t)$
23.	$\frac{s}{(s+a)(s+b)}$	$\frac{1}{a-b}(ae^{-at}-be^{-bt})$
24.	$\frac{\omega^2}{s(s^2+\omega^2)}$	$1-\cos\omega t$
25.	$\frac{s+\alpha}{s(s^2+\omega^2)}$	$\frac{\alpha}{\omega^2}-\frac{\sqrt{\alpha^2+\omega^2}}{\omega^2}\cos(\omega t+\varphi)$, $\varphi := \arg(\alpha+j\omega)$
25a.	$\frac{\omega_n^2(1+Ts)}{s(s^2+\omega_n^2)}$	$1-\sqrt{1+\omega_n^2T^2}\ \cos(\omega_n t+\varphi)$, $\varphi := \arg(1+j\omega_n T)$
26.	$\frac{1}{(s+a)(s^2+\omega^2)}$	$\frac{1}{a^2+\omega^2}e^{-at}+\frac{1}{\omega\sqrt{a^2+\omega^2}}\sin(\omega t-\varphi)$, $\varphi := \arg(\alpha+j\omega)$
26a.	$\frac{\omega_n^2}{(1+\tau s)(s^2+\omega_n^2)}$	$\frac{\omega_n^2\tau}{1+\omega_n^2\tau^2}e^{-\frac{t}{\tau}}+\frac{\omega_n}{\sqrt{1+\omega_n^2\tau^2}}\ \sin(\omega_n t-\varphi)$, $\varphi := \arg(1+j\omega_n\tau)$
27.	$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{-at}\sin bt$
27a.	$\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin\omega_n\sqrt{1-\delta^2}\ t$
28.	$\frac{s+\alpha}{(s+a)^2+b^2}$	$\frac{\sqrt{(\alpha-a)^2+b^2}}{b}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(\alpha-a+jb)$
28a.	$\frac{\omega_n^2(1+Ts)}{s^2+2\delta\omega_n s+\omega_n^2}$	$\omega_n\sqrt{\frac{1-2\delta\omega_n T+\omega_n^2T^2}{1-\delta^2}}e^{-\delta\omega_n t}\cdot$ $\sin\left(\omega_n\sqrt{1-\delta^2}\ t+\varphi\right)$, $\varphi := \arg\left(1-\delta\omega_n T+j\omega_n T\sqrt{1-\delta^2}\right)$
29.	$\frac{1}{s\left((s+a)^2+b^2\right)}$	$\frac{1}{a^2+b^2}-\frac{1}{b\sqrt{a^2+b^2}}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(a+jb)^{(1)}$
29a.	$\frac{\omega_n^2}{s(s^2+2\delta\omega_n s+\omega_n^2)}$	$1-\frac{1}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin\left(\omega_n\sqrt{1-\delta^2}\ t+\varphi\right)$, $\varphi := \arg\left(\delta+j\sqrt{1-\delta^2}\right)$

(1) Nel dedurre la relazione (29.) e altre successive si è fatto uso delle identità

$$-\arg(-\sigma+j\omega)=\arg(\sigma+j\omega)\pm\pi\ ,$$
$$\sin(\vartheta\pm\pi)=-\sin\vartheta\ .$$

	$F(s)$	$f(t)$
30.	$\dfrac{s+\alpha}{s\big((s+a)^2+b^2\big)}$	$\dfrac{\alpha}{a^2+b^2}-\dfrac{1}{b}\sqrt{\dfrac{(\alpha-a)^2+b^2}{a^2+b^2}}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(\alpha-a+jb)+\arg(a+jb)$
30a.	$\dfrac{\omega_n^2(1+Ts)}{s(s^2+2\delta\omega_n s+\omega_n^2)}$	$1-\sqrt{\dfrac{1-2\delta\omega_n T+\omega_n^2T^2}{1-\delta^2}}e^{-\delta\omega_n t}\cdot \sin\big(\omega_n\sqrt{1-\delta^2}t+\varphi\big)$, $\varphi := \arg\big(1-\delta\omega_n T+j\omega_n T\sqrt{1-\delta^2}\big)+\arg\big(\delta+j\sqrt{1-\delta^2}\big)$
31.	$\dfrac{1}{(s+c)\big((s+a)^2+b^2\big)}$	$\dfrac{1}{(c-a)^2+b^2}e^{-ct}+\dfrac{1}{b\sqrt{(c-a)^2+b^2}}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(c-a+jb)$
31a.	$\dfrac{\omega_n^2}{(1+\tau s)(s^2+2\delta\omega_n s+\omega_n^2)}$	$\dfrac{\omega_n^2\tau}{1-2\delta\omega_n\tau+\omega_n^2\tau^2}e^{-\frac{t}{\tau}}+\dfrac{1}{\sqrt{1-\delta^2}}\cdot \dfrac{\omega_n}{\sqrt{1-2\delta\omega_n\tau+\omega_n^2\tau^2}}e^{-\delta\omega_n t}\cdot \sin\big(\omega_n\sqrt{1-\delta^2}\,t+\varphi\big)$, $\varphi := \arg\big(1-\delta\omega_n\tau+j\omega_n\tau\sqrt{1-\delta^2}\big)$
32.	$\dfrac{1}{s(s+c)\big((s+a)^2+b^2\big)}$	$\dfrac{1}{c(a^2+b^2)}-\dfrac{1}{c\big((c-a)^2+b^2\big)}e^{-ct}-\dfrac{1}{b\sqrt{a^2+b^2}\sqrt{(c-a)^2+b^2}}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(a+jb)-\arg(c-a+jb)$
32a.	$\dfrac{\omega_n^2}{s(1+\tau s)(s^2+2\delta\omega_n s+\omega_n^2)}$	$1-\dfrac{\omega_n^2\tau^2}{1-2\delta\omega_n\tau+\omega_n^2\tau^2}e^{-\frac{t}{\tau}}-\dfrac{1}{\sqrt{1-\delta^2}}\cdot \dfrac{1}{\sqrt{1-2\delta\omega_n\tau+\omega_n^2\tau^2}}e^{-\delta\omega_n t}\cdot \sin\big(\omega_n\sqrt{1-\delta^2}\,t+\varphi\big)$, $\varphi := \arg\big(\delta+j\sqrt{1-\delta^2}\big)-\arg\big(1-\delta\omega_n\tau+j\omega_n\tau\sqrt{1-\delta^2}\big)$
33.	$\dfrac{s+\alpha}{s(s+c)\big((s+a)^2+b^2\big)}$	$\dfrac{\alpha}{c(a^2+b^2)}+\dfrac{c-\alpha}{c\big((c-a)^2+b^2\big)}e^{-ct}-\dfrac{\sqrt{(\alpha-a)^2+b^2}}{b\sqrt{a^2+b^2}\sqrt{(c-a)^2+b^2}}e^{-at}\sin(bt+\varphi)$, $\varphi := \arg(\alpha-a+jb)+\arg(a+jb)-\arg(c-a+jb)$

	$F(s)$	$f(t)$
33a.	$\dfrac{\omega_n^2(1+Ts)}{s(1+\tau s)(s^2+2\delta\omega_n s+\omega_n^2)}$	$1+\dfrac{\omega_n^2\tau(T-\tau)}{1-2\delta\omega_n\tau+\omega_n^2\tau^2}e^{-\frac{t}{\tau}}-\dfrac{1}{\sqrt{1-\delta^2}}\cdot$ $\sqrt{\dfrac{1-2\delta\omega_n T+\omega_n^2T^2}{1-2\delta\omega_n\tau+\omega_n^2\tau^2}}e^{-\delta\omega_n t}\cdot$ $\sin\left(\omega_n\sqrt{1-\delta^2}\,t+\varphi\right),$ $\varphi:=\arg\left(1-\delta\omega_n T+j\omega_n T\sqrt{1-\delta^2}\right)+$ $\arg\left(\delta+j\sqrt{1-\delta^2}\right)-$ $\arg\left(1-\delta\omega_n\tau+j\omega_n\tau\sqrt{1-\delta^2}\right)$
34.	$\dfrac{1}{s^2(s+a)}$	$\dfrac{1}{a^2}(at-1+e^{-at})$
34a.	$\dfrac{1}{s^2(1+\tau s)}$	$t-\tau\left(1-e^{-\frac{t}{\tau}}\right)$
35.	$\dfrac{s+\alpha}{s^2(s+a)}$	$\dfrac{1}{a^2}\left(\alpha at+(a-\alpha)(1-e^{-at})\right)$
35a.	$\dfrac{1+Ts}{s^2(1+\tau s)}$	$t+(T-\tau)\left(1-e^{-\frac{t}{\tau}}\right)$
36.	$\dfrac{1}{s(s+a)^2}$	$\dfrac{1}{a^2}\left(1-(1+at)e^{-at}\right)$
36a.	$\dfrac{1}{s(1+\tau s)^2}$	$1-\dfrac{\tau+t}{\tau}e^{-\frac{t}{\tau}}$
37.	$\dfrac{s+\alpha}{s(s+a)^2}$	$\dfrac{1}{a^2}\left(\alpha+\left(a(a-\alpha)t-\alpha\right)e^{-at}\right)$
37a.	$\dfrac{1+Ts}{s(1+\tau s)^2}$	$1+\left(\dfrac{T-\tau}{\tau^2}t-1\right)e^{-\frac{t}{\tau}}$