Kruskal's algorithm

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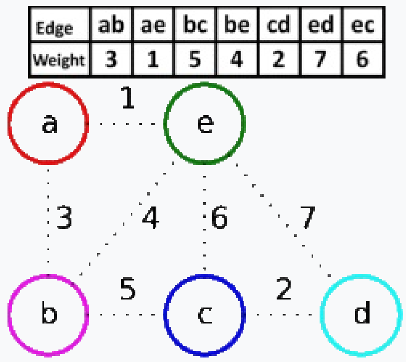
Kruskal's algorithm is a minimum-spanning-tree algorithm which finds an edge of the least possible weight that connects any two trees in the forest.[1] It is a greedy algorithm in graph theory as it finds a minimum spanning tree for a connected weighted graph adding increasing cost arcs at each step.[1] This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a *minimum spanning forest* (a minimum spanning tree for each connected component).

This algorithm first appeared in *Proceedings of the American Mathematical Society*, pp. 48–50 in 1956, and was written by Joseph Kruskal.[2]

Other algorithms for this problem include Prim's algorithm, Reverse-delete algorithm, and Borůvka's algorithm.

Kruskal's algorithm



Class	Minimum spanning tree
Data structure	Disjoint-set data structure
Average performance	$O(\|E\|\log\|V\|)$
Worst-case space complexity	$\Omega(\|E\|+\|V\|)$

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Algorithm

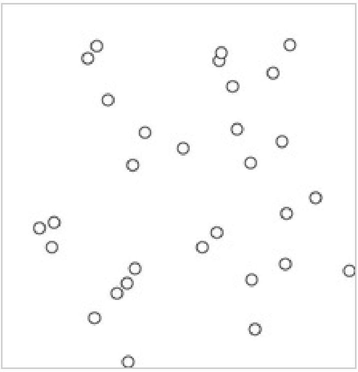
- create a forest F (a set of trees), where each vertex in the graph is a separate tree
- create a set S containing all the edges in the graph
- while S is nonempty and F is not yet spanning
  - remove an edge with minimum weight from S
  - if the removed edge connects two different trees then add it to the forest F, combining two trees into a single tree

At the termination of the algorithm, the forest forms a minimum spanning forest of the graph. If the graph is connected, the forest has a single component and forms a minimum spanning tree

Pseudocode

The following code is implemented with disjoint-set data structure:

```
KRUSKAL(G):
A = ∅
foreach v ∈ G.V:
   MAKE-SET(v)
foreach (u, v) in G.E ordered by weight(u, v), increasing:
   if FIND-SET(u) ≠ FIND-SET(v):
      A = A ∪ {(u, v)}
      UNION(u, v)
return A
```



A demo for Kruskal's algorithm based on Euclidean distance.

Complexity

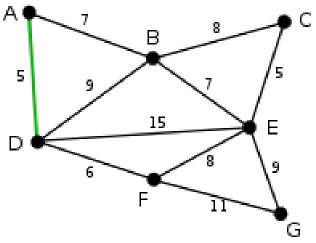
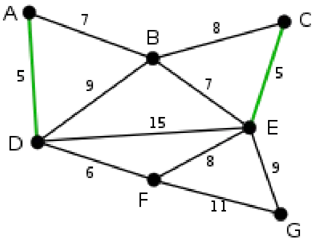
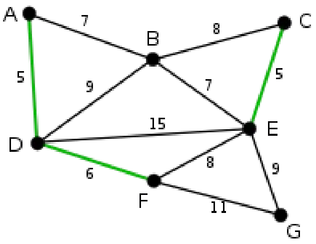
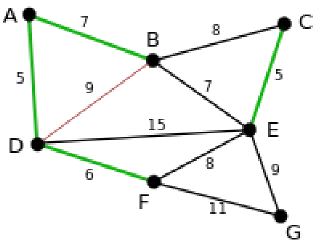
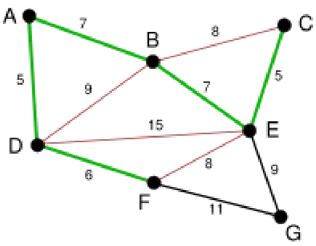
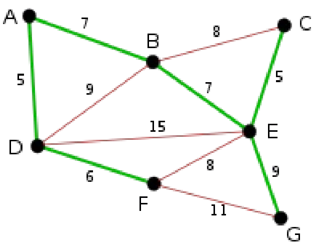
Kruskal's algorithm can be shown to run in $O(E \log E)$ time, or equivalently, $O(E \log V)$ time, where E is the number of edges in the graph and V is the number of vertices, all with simple data structures. These running times are equivalent because:

- E is at most V^2 and $\log V^2 = 2\log V$ is $O(\log V)$.
- Each isolated vertex is a separate component of the minimum spanning forest. If we ignore isolated vertices we obtain $V \leq 2E$, so $\log V$ is $O(\log E)$.

We can achieve this bound as follows: first sort the edges by weight using a comparison sort in $O(E \log E)$ time; this allows the step "remove an edge with minimum weight from S" to operate in constant time. Next, we use a disjoint-set data structure (Union&Find) to keep track of which vertices are in which components. We need to perform $O(V)$ operations, as in each iteration we connect a vertex to the spanning tree, two 'find' operations and possibly one union for each edge. Even a simple disjoint-set data structure such as disjoint-set forests with union by rank can perform $O(V)$ operations in $O(V \log V)$ time. Thus the total time is $O(E \log E) = O(E \log V)$.

Provided that the edges are either already sorted or can be sorted in linear time (for example with counting sort or radix sort), the algorithm can use a more sophisticated disjoint-set data structure to run in $O(E\ \alpha(V))$ time, where α is the extremely slowly growing inverse of the single-valued Ackermann function.

Example

Image	Description
	**AD** and **CE** are the shortest edges, with length 5, and **AD** has been arbitrarily chosen, so it is highlighted.
	**CE** is now the shortest edge that does not form a cycle, with length 5, so it is highlighted as the second edge.
	The next edge, **DF** with length 6, is highlighted using much the same method.
	The next-shortest edges are **AB** and **BE**, both with length 7. **AB** is chosen arbitrarily, and is highlighted. The edge **BD** has been highlighted in red, because there already exists a path (in green) between **B** and **D**, so it would form a cycle (**ABD**) if it were chosen.
	The process continues to highlight the next-smallest edge, **BE** with length 7. Many more edges are highlighted in red at this stage: **BC** because it would form the loop **BCE**, **DE** because it would form the loop **DEBA**, and **FE** because it would form **FEBAD**.
	Finally, the process finishes with the edge **EG** of length 9, and the minimum spanning tree is found.

Proof of correctness

The proof consists of two parts. First, it is proved that the algorithm produces a spanning tree. Second, it is proved that the constructed spanning tree is of minimal weight.

Spanning tree

Let P be a connected, weighted graph and let Y be the subgraph of P produced by the algorithm. Y cannot have a cycle, being within one subtree and not between two different trees. Y cannot be disconnected, since the first encountered edge that joins two components of Y would have been added by the algorithm. Thus, Y is a spanning tree of P.

Minimality

We show that the following proposition $\boldsymbol{P}$ is true by induction: If F is the set of edges chosen at any stage of the algorithm, then there is some minimum spanning tree that contains F.

- Clearly $\boldsymbol{P}$ is true at the beginning, when F is empty: any minimum spanning tree will do, and there exists one because a weighted connected graph always has a minimum spanning tree.
- Now assume $\boldsymbol{P}$ is true for some non-final edge set F and let T be a minimum spanning tree that contains F. If the next chosen edge e is also in T, then $\boldsymbol{P}$ is true for $F + e$. Otherwise, $T + e$ has a cycle C and there is another edge f that is in C but not F. (If there were no such edge f, then e could not have been added to F, since doing so would have created the cycle C.) Then $T - f + e$ is a tree, and it has the same weight as T, since T has minimum weight and the weight of f cannot be less than the weight of e, otherwise the algorithm would have chosen f instead of e. So $T - f + e$ is a minimum spanning tree containing $F + e$ and again $\boldsymbol{P}$ holds.
- Therefore, by the principle of induction, $\boldsymbol{P}$ holds when F has become a spanning tree, which is only possible if F is a minimum spanning tree itself.

See also

- Dijkstra's algorithm
- Prim's algorithm
- Borůvka's algorithm
- Reverse-delete algorithm
- Single-linkage clustering

References

1. Cormen, Thomas; Charles E Leiserson, Ronald L Rivest, Clifford Stein (2009). *Introduction To Algorithms* (Third ed.). MIT Press. p. 631. ISBN 0262258102.
2. Kruskal, J. B. (1956). "On the shortest spanning subtree of a graph and the traveling salesman problem". *Proceedings of the American Mathematical Society*. **7**: 48–50. JSTOR 2033241 (https://www.jstor.org/stable/2033241). doi:10.1090/S0002-9939-1956-0078686-7 (https://doi.org/10.1090%2FS0002-9939-1956-0078686-7).

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*, Second Edition. MIT Press and McGraw-Hill, 2001. ISBN 0-262-03293-7. Section 23.2: The algorithms of Kruskal and Prim, pp. 567–574.
- Michael T. Goodrich and Roberto Tamassia. *Data Structures and Algorithms in Java*, Fourth Edition. John Wiley & Sons, Inc., 2006. ISBN 0-471-73884-0. Section 13.7.1: Kruskal's Algorithm, pp. 632..

External links

- Krushkal's Algorithm with example and program in c++ (http://thecodersportal.com/mst-krushkal/)
- How does Kruskal Algorithm progress? Animation (https://www.youtube.com/watch?v=o8Sqm1_3BRo)