Soluzione esercizi sulla convoluzione -

Esercizio (1) conv.

Si calcoli $y(t) = x(t) \otimes h(t)$ essendo:

$$x(t) = e^{-\alpha t} u(t) \qquad \text{con } \alpha, \beta > 0$$
$$h(t) = e^{-\beta t} u(t)$$

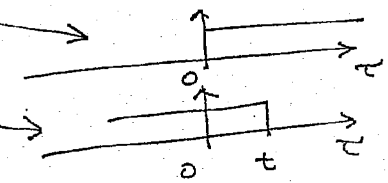
Si distinguano i casi: $\alpha \neq \beta$, $\alpha = \beta$

È, per definizione:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} e^{-\alpha\tau} u(\tau)\, e^{-\beta(t-\tau)} u(t-\tau)\, d\tau$$

Il prodotto $u(\tau) \cdot u(t-\tau)$ vale:

$$u(\tau) \cdot u(t-\tau) = \begin{cases} 1 & \text{per } t>0,\ 0<\tau<t \\ 0 & \text{" } t>0;\ \tau>t,\ \tau<0 \\ 0 & \text{" } t<0 \text{ per } \forall \tau \end{cases}$$



ciò permette di sostituire le $u(\cdot)$ con opportuni limiti di integrazione-

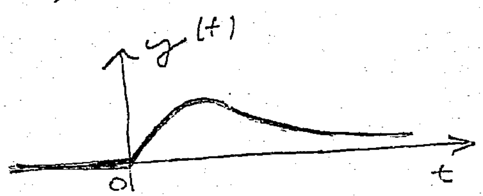
Quindi, $\boxed{\text{per } t>0:}$ e $(\alpha \neq \beta)$:

$$y(t) = \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)}\, d\tau = \int_0^t e^{-\alpha\tau} e^{-\beta t} e^{\beta\tau}\, d\tau =$$

$$= e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau}\, d\tau = \frac{e^{-\beta t}}{(\beta-\alpha)} \left[e^{+(\beta-\alpha)\tau} \right]_0^t =$$

$$= \frac{e^{-\beta t}}{(\beta-\alpha)} \left[e^{(\beta-\alpha)t} - 1 \right] = \frac{e^{-\alpha t} - e^{-\beta t}}{(\beta-\alpha)}$$

e: $y(t) = 0$ $\boxed{\text{per } t<0}$

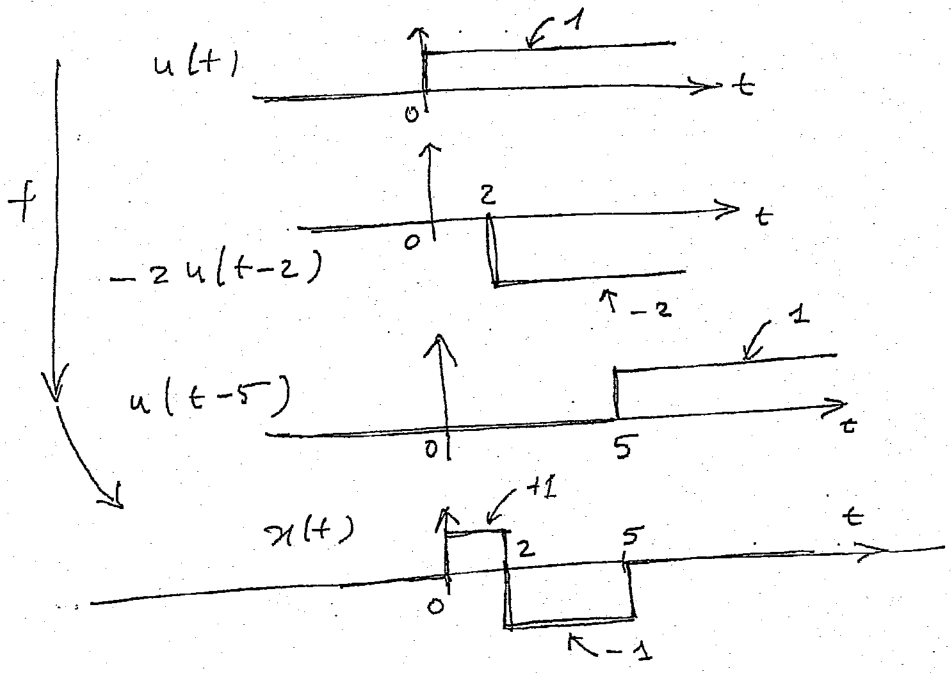


Esercizio (2) CONV.

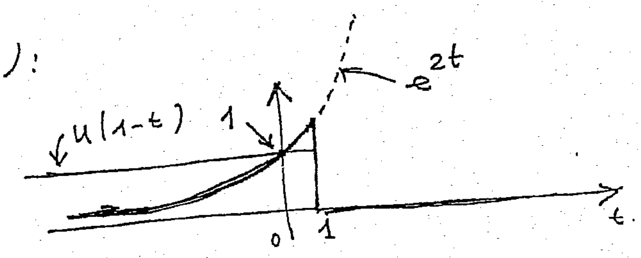
Si calcoli la convoluzione $y(t) = x(t) \otimes h(t)$ essendo:

$$x(t) = u(t) - 2u(t-2) + u(t-5)$$
$$h(t) = e^{2t}\, u(1-t).$$

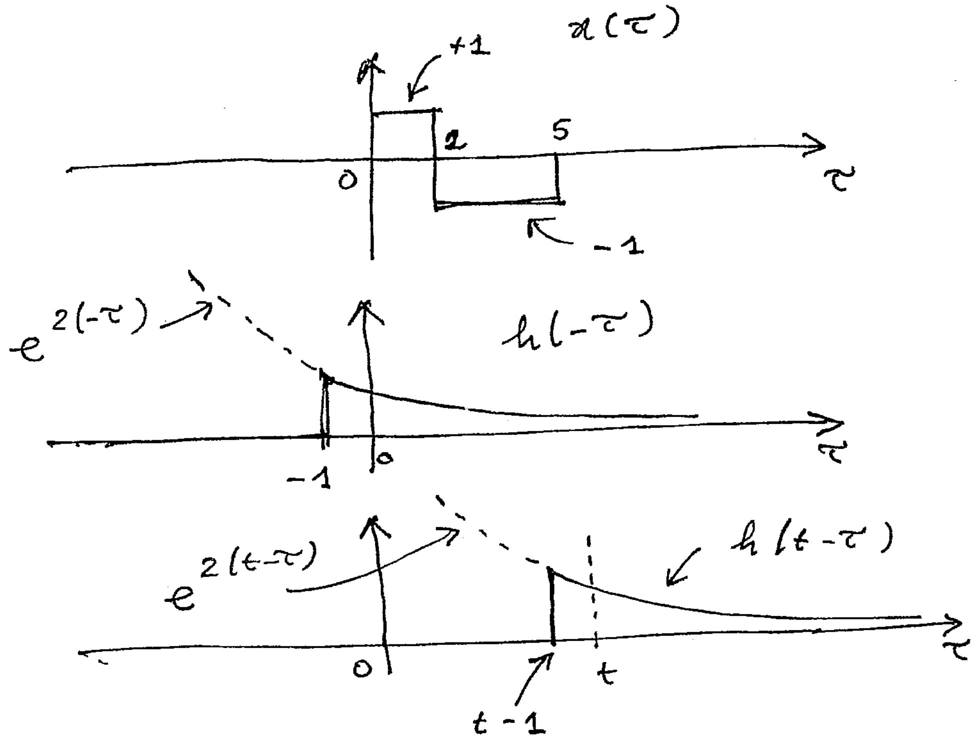
Vediamo come è fatto $x(t)$:



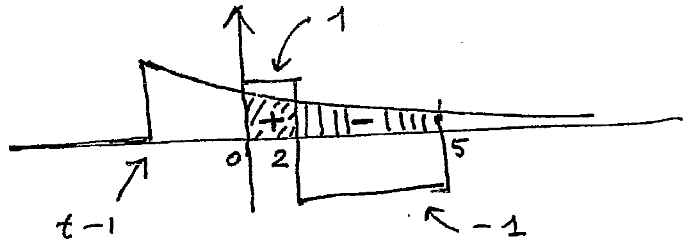
E $h(t)$:



C3



- Situazione per $-\infty < t-1 < 0$ ossia per $-\infty < t < 1$:

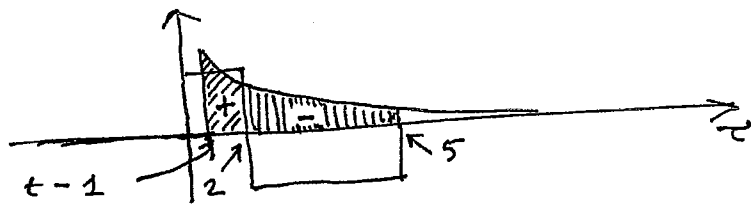


$$y_1(t) = \int_0^2 e^{2(t-\tau)}\,d\tau - \int_2^5 e^{2(t-\tau)}\,d\tau =$$

$$= e^{2t}\left\{\left[\frac{e^{-2\tau}}{-2}\right]_0^2 - \left[\frac{e^{-2\tau}}{-2}\right]_2^5\right\} =$$

$$\boxed{= \frac{e^{2t}}{-2}\left\{e^{-4} - 1 - e^{-10} + e^{4}\right\}}$$ per $-\infty < t < 1$

Situazione per $0 < t-1 < 2$ ossia per $1 < t < 3$



(c4)

Quindi:

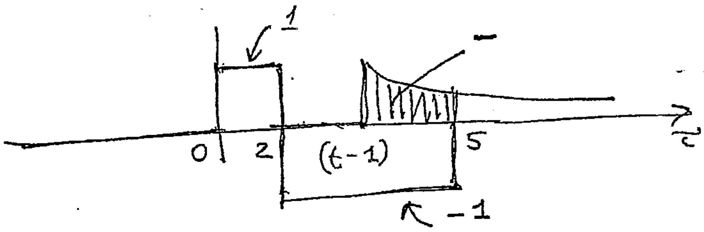
$$\boxed{y_2(t)} = \int_{t-1}^{2} e^{2(t-\tau)}\,d\tau - \int_{2}^{5} e^{2(t-\tau)}\,d\tau =$$

$$= e^{2t}\left\{\left[\frac{e^{-2\tau}}{-2}\right]_{t-1}^{2} - \left[\frac{e^{-2\tau}}{-2}\right]_{2}^{5}\right\} =$$

$$= \boxed{\frac{e^{2t}}{-2}\left\{e^{-4} - e^{-2(t-1)} - e^{-10} + e^{-4}\right\}} \quad \text{per } 1<t<3$$

$\leftarrow y_2(t)$

Per $2<t-1<5$ ovvero $\boxed{3<t<6}$



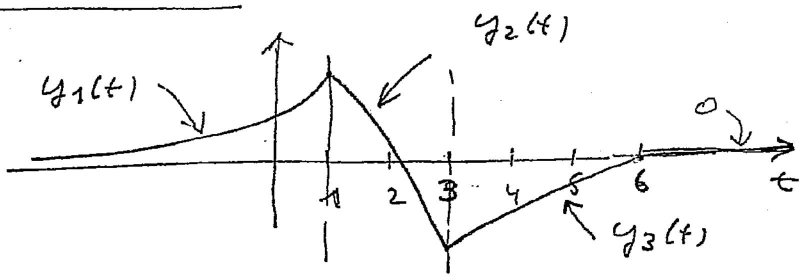
$$\boxed{y_3(t)} = -\int_{t-1}^{5} e^{2(t-\tau)}\,d\tau =$$

$$= -e^{2t}\int_{t-1}^{5} e^{-2\tau}\,d\tau = -e^{2t}\left\{\left[\frac{e^{-2\tau}}{-2}\right]_{t-1}^{5}\right\} =$$

$$= \boxed{\frac{e^{2t}}{2}\left\{e^{-10} - e^{-2(t-1)}\right\}} \quad \text{per } 3<t<6$$

$\leftarrow y_3(t)$

Risultato:

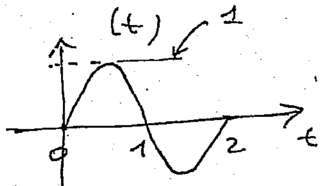


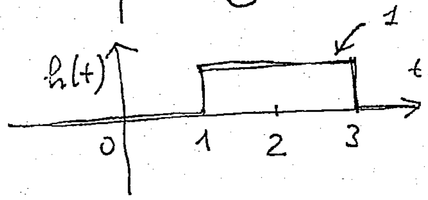
Esercizio 3 CONV.

Si calcoli la convoluzione $y(t)=x(t)\otimes h(t)$

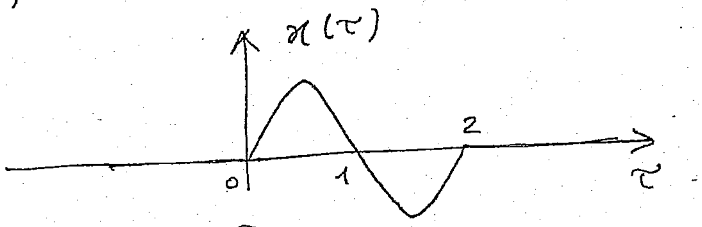
essendo:

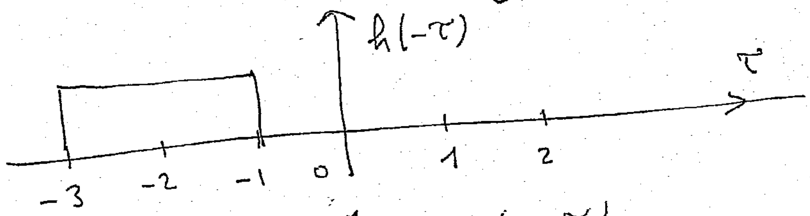
$$\begin{cases} x(t) = (\sin \pi t)\left[u(t)-u(t-2)\right] \\ h(t) = u(t-1)-u(t-3) \end{cases}$$

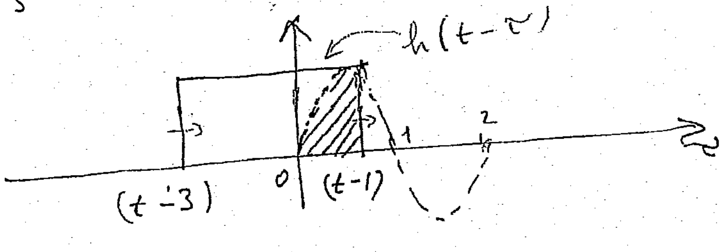




Graficamente, per trovare i limiti di integrazione:





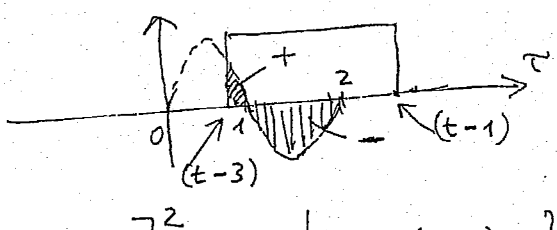


Quindi:

Per $0<t-1<2$ ovvero $\boxed{1<t<3}$

$$y(t)=\int_0^{(t-1)} \sin\pi\tau \, d\tau = \left[\frac{-\cos\pi\tau}{\pi}\right]_0^{t-1} = \frac{1}{\pi}\underbrace{\left\{\overset{\cos 0}{1}-\cos\left[\pi(t-1)\right]\right\}}_{y_1(t)}$$

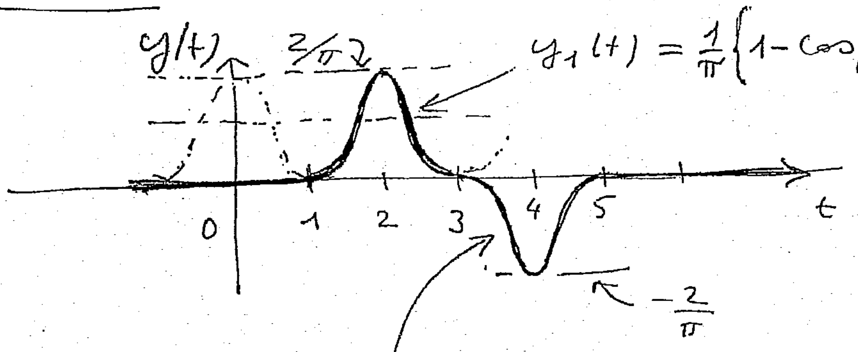
Per $0<t-3<2$ ovvero $\boxed{3<t<5}$



$$y(t)=\int_{t-3}^{2} \sin\pi\tau \, d\tau = \left[\frac{-\cos\pi\tau}{\pi}\right]_{t-3}^{2} = \underbrace{\frac{1}{\pi}\left\{\cos\pi(t-3)\underset{-\cos 2\pi}{-1}\right\}}_{y_2(t)}$$

./.

Risultato:



$$y_1(t) = \frac{1}{\pi}\left\{1-\cos\left[\pi(t-1)\right]\right\}$$

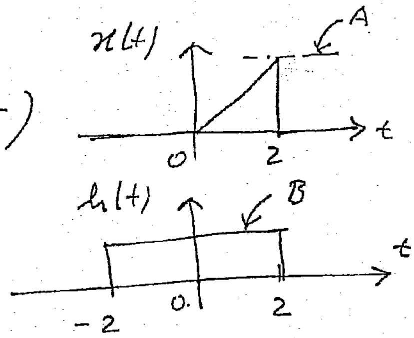
$$y_2(t) = \frac{1}{\pi}\left\{\cos\left[\pi(t-3)\right]-1\right\}$$

Esercizio (4) CONV.

Si calcoli la convoluzione: $y(t) = x(t) \otimes h(t)$

essendo:

$$\begin{cases} x(t) = \frac{A}{2}\, t \cdot \mathrm{rect}\left(\frac{t-1}{2}\right) \\ h(t) = B \cdot \mathrm{rect}\left(\frac{t}{4}\right) \end{cases}$$

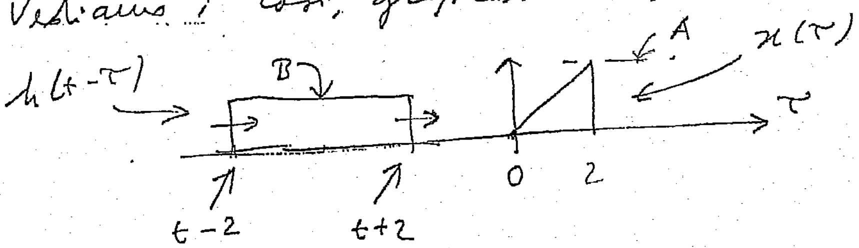


Dalla definizione:

$$y(t) = \int_{-\infty}^{+\infty} \frac{A}{2} \tau\, \mathrm{rect}\left(\frac{\tau-1}{2}\right) \cdot B\, \mathrm{rect}\left(\frac{t-\tau}{4}\right) d\tau \quad \#$$

Il prodotto delle $\mathrm{rect}(\cdot)$ vale 1 solo in alcuni intervalli che determinano i limiti di integrazione.

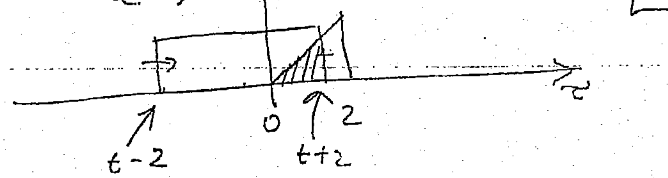
Vediamo i casi graficamente:



Vari casi:

I°) $y(t) = 0$ per $t+2<0$ ossia $\boxed{t<-2}$ e per $t-2>2$ ossia $\boxed{t>4}$

II°)



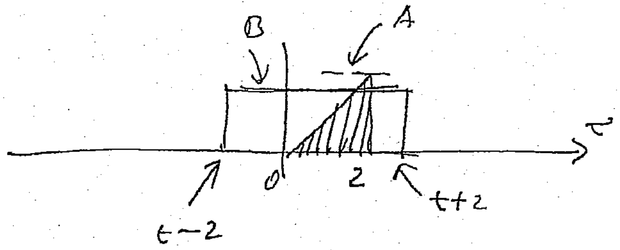
$$\boxed{y(t) = \int_0^{t+2} \frac{AB}{2} \tau\, d\tau = \frac{AB}{2}\left[\frac{\tau^2}{2}\right]_0^{t+2} = \frac{AB}{4}\left[(t+2)^2 - 0\right] = \frac{AB}{4}(t+2)^2}$$

$y_1(t)$

per $0<t+2<2$ ossia $\boxed{-2<t<0}$

(2

III)

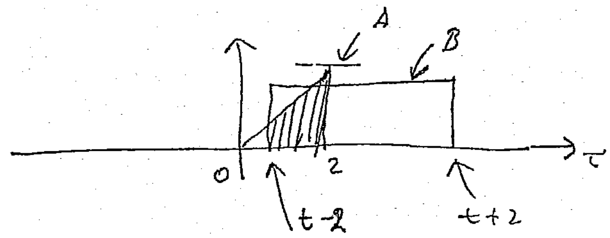


$$\boxed{y(t) = \int_0^2 \frac{AB}{2}\tau\, d\tau = \frac{AB \cdot 2}{2} = AB} \leftarrow y_2(t)$$

per $\quad 2 < t+2 < 4$

ossia $\quad \boxed{0 < t < 2}$

IV)



$$\boxed{y(t) = \int_{t-2}^{2} \frac{AB}{2}\, d\tau = \frac{AB}{2}\left[\frac{\tau^2}{2}\right]_{t-2}^{2} = \frac{AB}{4}\left[4 - (t-2)^2\right]} \leftarrow y_3(t)$$

per $\quad 0 < t-2 < 2$

ossia $\quad \boxed{2 < t < 4}$

In definitiva: