Esercizi sulla Serie di Fourier.

$\boxed{\text{SOLUZIONE}}$

1) Sia $y(t) = x(t-t_0)$ - I coefficienti Y_k di $y(t)$ sono:

$$Y_k = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-j2\pi k f_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x(t-t_0)\, e^{-j2\pi k f_0 t}\, dt$$

si ponga $\lambda \triangleq t-t_0$ da cui $t=\lambda+t_0$ e $dt = d\lambda$, quindi:

$$Y_k = \frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-j2\pi k f_0 t_0}\, e^{-j2\pi k f_0 \lambda}\, d\lambda =$$

$$= e^{-j2\pi k f_0 t_0}\cdot \underbrace{\frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-j2\pi k f_0 \lambda}\, d\lambda}_{X_k} = \boxed{X_k \cdot e^{-j2\pi k f_0 t_0}}$$

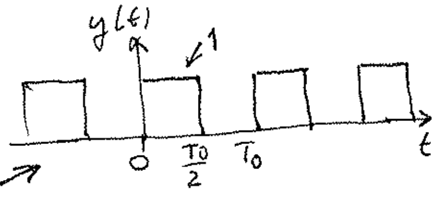
2) Si parte dai coefficienti di $x(t)$:

$$X_k = \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi k f_0 t}\, dt = \frac{1}{T_0}\left[\frac{e^{-j2\pi k f_0 t}}{-j2\pi k f_0}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} =$$

$$= \frac{\tau}{\cancel{T_0}}\,\frac{\operatorname{sen} \pi k f_0 \tau}{\pi \cancel{k}\cancel{f_0}\tau} = \boxed{\frac{\tau}{T_0}\operatorname{sinc}(k f_0 \tau)}$$

Siccome è: $y(t) = x\left(t-\frac{\tau}{2}\right)$ da cui (Vedi es. n. 1)

$$\boxed{Y_k = \frac{\tau}{T_0}\operatorname{sinc}(k f_0 \tau)\cdot e^{-j2\pi k f_0 \frac{\tau}{2}}}$$



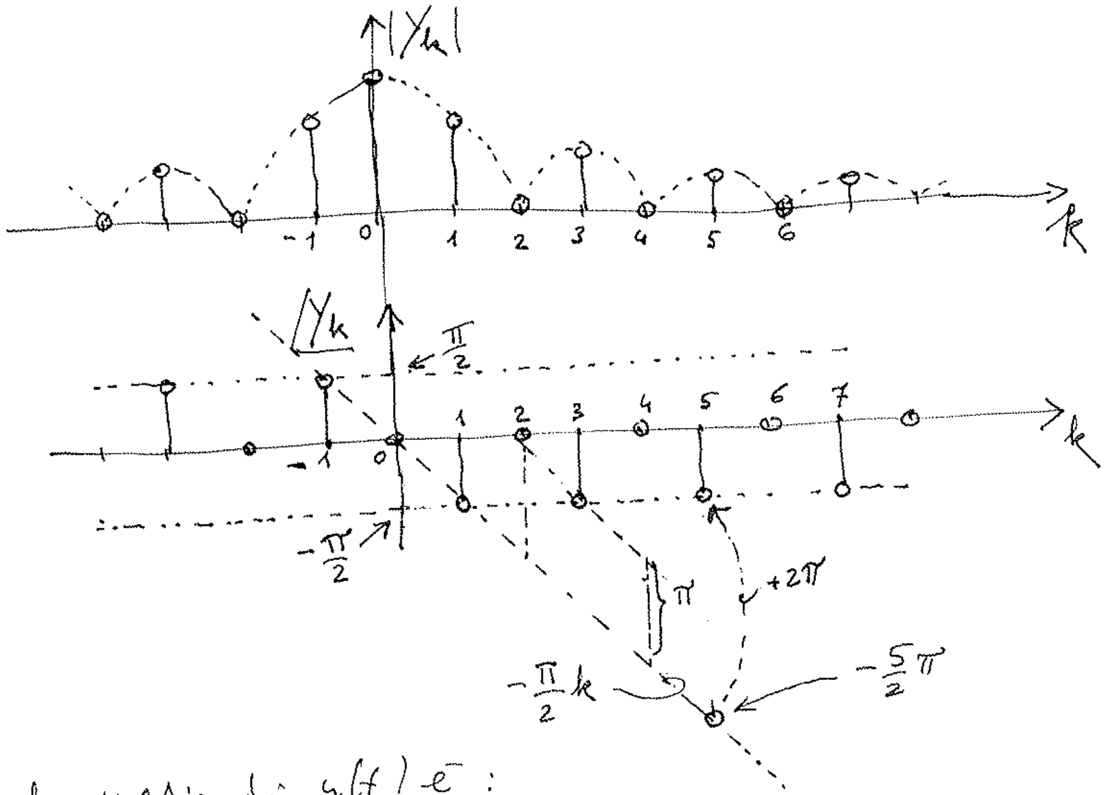
3) Dalla precedente, ponendo $\tau = \frac{T_0}{2}$ si ha:

$$\boxed{Y_k = \frac{1}{2}\operatorname{sinc}\left(k \cancel{f_0}\frac{\cancel{T_0}}{2}\right)\cdot e^{-j\cancel{2}\pi k \cancel{f_0}\frac{\cancel{T_0}}{\cancel{4}\,2}} = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right) e^{-j\pi\frac{k}{2}}}$$

Da cui: $|Y_k| = \frac{1}{2}\left|\operatorname{sinc}\left(\frac{k}{2}\right)\right|$ ← $= 0$ per k pari.

$$\angle Y_k = \angle \operatorname{sinc}\left(\frac{k}{2}\right) - \pi\frac{k}{2}$$

$$\angle \operatorname{sinc}\left(\frac{k}{2}\right) = \begin{cases} 0 & \text{per } \operatorname{sinc}\left(\frac{k}{2}\right) \geq 0 \\ \pm\pi & \text{per } \operatorname{sinc}\left(\frac{k}{2}\right) < 0 \end{cases}$$



4) Il valor medio di $y(t)$ è:

$$\langle y(t)\rangle = \frac{1}{T_0}\int_{T_0} y(t)\,dt = \frac{1}{T_0}\left(\frac{T_0}{2}\right) = \frac{1}{2} \quad \text{(uguale anche a } Y_0\text{)}$$

5)

Si applichi la definizione:

$$\boxed{X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt} = \frac{1}{T_0}\int_0^{T_0} \underbrace{\left(\frac{A}{T_0}t\right)}_{x(t),\ \text{in } 0<t<T_0} \cdot e^{-j2\pi k f_0 t}\, dt =$$

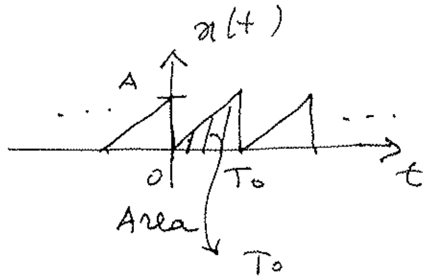
(integrazione per parti)

$$= \frac{A}{T_0^2}\left\{\left[\frac{t\cdot e^{-j2\pi k f_0 t}}{-j2\pi k f_0}\right]_0^{T_0} - \int_0^{T_0}\frac{e^{-j2\pi k f_0 t}}{-j2\pi k f_0}\,dt\right\} =$$

$$= \frac{A}{T_0^2}\left\{\left[\frac{T_0\cdot \overbrace{e^{-j2\pi k \cancel{f_0 T_0}}}^{1} - 0}{-j2\pi k f_0}\right] - \left[\frac{e^{-j2\pi k f_0 t}}{(-j2\pi k f_0)^2}\right]_0^{T_0}\right\} =$$

$$= \frac{A}{T_0^2}\left\{\left[\frac{T_0}{-j2\pi k f_0}\right] - \underbrace{\left[\frac{\overbrace{e^{-j2\pi k f_0 T_0}}^{1} - 1}{(-j2\pi k f_0)^2}\right]}_{=0}\right\} =$$

$$= \frac{A}{\cancel{T_0}} \frac{1}{-j 2\pi k \cancel{f_0}} = \boxed{j \frac{A}{2\pi k}}$$



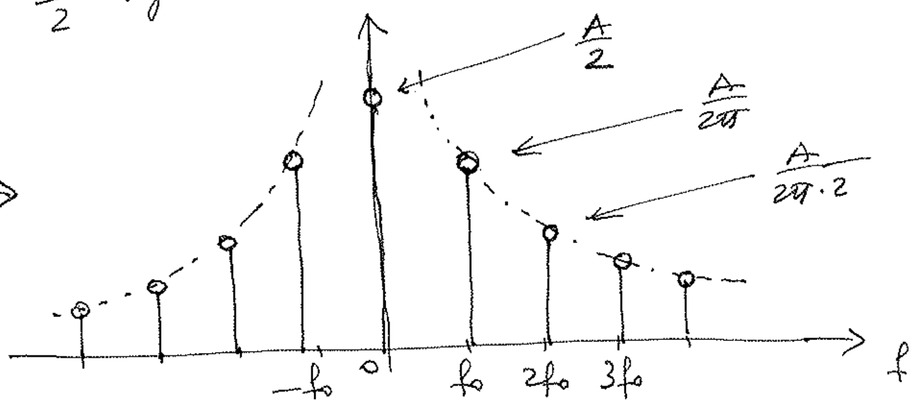
Quindi:

$$\boxed{X_k = \begin{cases} \dfrac{A}{2} & \text{per } k = 0 \\[2mm] j\,\dfrac{A}{2\pi k} & \text{per } k \neq 0 \end{cases}} \leftarrow \left(X_0 = \langle x(t) \rangle = \frac{1}{T_0} \int_0^{T_0} \left(\frac{A}{T_0} t \right) dt = \frac{1}{T_0} \frac{A T_0}{2} = \frac{A}{2} \right)$$

Da cui:

$$\begin{cases} |X_k| = \begin{cases} \dfrac{A}{2} & \text{per } k = 0 \\[2mm] \dfrac{A}{2\pi |k|} & \text{per } k \neq 0 \end{cases} \\[6mm] \angle X_k = \dfrac{\pi}{2} \operatorname{sgn}(k) \end{cases}$$

$|X_k| \longrightarrow$



$\angle X_k \longrightarrow$